ON HILL'S METHOD IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

(K METODU KHILLA V TEORII LINEINYKH DIFFERENTSIAL'NYKH URAVNENII S PERIODICHESKIMI KOEFFITSIENTAMI)

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K. G. VALEEV
(Leningrad)

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We consider a certain class of systems of linear differential equations with periodic coefficients which have the property that, by means of the Laplace transformation, they may be converted to a system of linear difference equations. The latter may be solved by the method of infinite determinants, i.e. the method of Hill [1-7]. We consider the convergence of a certain type of infinite determinants which are not normal [3]. In the particular case of a single differential equation with sinusoidal coefficients we construct the Laplace transform of the solution by means of continued fractions [8]. We study the dynamical stability of the solutions of certain differential equations which occur in engineering [6].

Capital letters denote matrices and vectors, while lower-case letters refer to scalars. A matrix which depends upon a certain variable will be said to be holomorphic (or bounded, etc.) in a certain domain provided that each element of the matrix is holomorphic (or bounded, etc.) in this domain. By the application of the Laplace transformation [9] to the matrix $Y(t)$ $(t \geqslant 0)$ is meant the application of the Laplace transformation to each of the elements of the matrix $Y(t)$ $(t \geqslant 0)$. The correspondence between the original matrix $Y(t)$ $(t \geqslant 0)$ and its transform $F(p)$, supposed to be continued analytically to its entire domain of existence, will be denoted simply by

$$Y(t) \doteq F(p) \qquad (0.1)$$

1. Consider a system of linear differential equations with periodic coefficients of the special form

$$\sum_{q=-l}^{l} e^{-\omega q t} L_q(d) Y(t) = \Phi(t) \tag{1.1}$$

Here $Y(t)$ is an m-dimensional vector, $\omega \neq 0$ is a purely imaginary number, l is a finite number, $L_q(d)$ is a linear differential operator

$$L_q(d) = \sum_{j=0}^{n} A_{qj} d^j \qquad \left(d = \frac{d}{dt}\right) \tag{1.2}$$

and the A_{qj} are constant, complex, m-by-m matrices such that

$$A_{qj} = \| a_{sr}^{qj} \|_1^m \quad (q = 0, \pm 1, \ldots, \pm l,\ j = 0, 1, \ldots, n), \quad A_{0n} = E \ldots A_{qn} \equiv 0 \quad (q \neq 0) \tag{1.3}$$

with the matrix E being the identity matrix. Let us seek the solution $Y(t)$ of the system (1.1) satisfying the initial conditions

$$Y(0) = Y_0^{(0)}, \ldots, Y^{(n-1)}(0) = Y_0^{(n-1)} \tag{1.4}$$

In order to do this, let us suppose that $\Phi(t)$ $(t \geqslant 0)$ possesses a Laplace transform $Q(p)$ which is regular and bounded for $\operatorname{Re} p \geqslant b$.

Applying the Laplace transformation [9] to the system (1.1) for $t \geqslant 0$, and using (0.1), we obtain for $F(p)$ the system of linear difference equations

$$\sum_{q=-l}^{l} L_q(p + \omega q) F(p + \omega q) = R(p) \tag{1.5}$$

where

$$R(p) \equiv Q(p) + \sum_{q=-l}^{l} \Psi_q(p + \omega q), \qquad \Psi_q(p) \equiv \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} A_{qk} Y_0^{(j)} p^{k-j-1} \tag{1.6}$$

Let us seek a solution $F(p)$ of the system of difference equations (1.5) which is regular and bounded for $\operatorname{Re} p \geqslant b_1$ (b_1 = const). Such a solution, for example, is the transform of $Y(t)$ itself. Replacing $p + \omega k$ by p in (1.5), and then dividing through by $(k\omega)^n$ $(k \neq 0)$, we obtain instead of (1.5) the following infinite system of linear algebraic equations in the quantities $F(p + \omega k)$ $(k = 0, \pm 1, \pm 2, \ldots)$:

$$\sum_{q=-l}^{l} (k\omega)^{-n} L_q(p + \omega(k + q)) F(p + \omega(k + q)) = (k\omega)^{-n} R(p + \omega k) \qquad (k \neq 0) \tag{1.7}$$

where p may be considered as a parameter in the coefficients. Further, in order not to have to consider the special case $k = 0$ in particular, let us include (1.5) in (1.7) by agreeing to regard $(k\omega)^{-n}$ as equal to one when $k = 0$.

The matrix determinant of the system (1.7) is itself a quasimatrix with quasimatrix elements $((k - q)\omega)^{-n}L_q(p + \omega k)$. This determinant, which will be denoted by $\Delta(p)$, has the form

$$\Delta(p) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^{-n} L_0(p+\omega) & \omega^{-n} L_{-1}(p) & \omega^{-n} L_{-2}(p-\omega) & \cdot \\ \cdot & L_1(p+\omega) & L_0(p) & L_{-1}(p-\omega) & \cdot \\ \cdot & (-\omega)^{-n} L_2(p+\omega) & (-\omega)^{-n} L_1(p) & (-\omega)^{-n} L_0(p-\omega) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \tag{1.8}$$

2. Consider the sequence of determinants $\Delta_\gamma(p)$ $(\gamma = 1, 2, \ldots)$, containing $(2\gamma - 1)m$ rows and columns, whose matrix is obtained from the matrix of the matrix determinant $\Delta(p)$ in such a way that the quasi-element $L_0(p)$ lies at the center of the determinant $\Delta_\gamma(p)$.

Suppose that Σ is a finite domain of the plane of the complex variable p. Consider the convergence in Σ of the determinant $\Delta(p)$ and of the algebraic complements of the elements of the column which contains $L_0(p)$; that is, the convergence of the determinants $\Delta_\gamma(p)$ as $\gamma \to \infty$, and the convergence of the algebraic complements of the corresponding elements occurring in $\Delta_\gamma(p)$ (γ is supposed to be sufficiently large). Let us denote the elements of the determinant $\Delta(p)$ by $c_{rs}{}^{qk}(p)$, where the subscripts and superscripts mean that the element $c_{rs}{}^{qk}(p)$ lies on the intersection of row s and column r of the quasi-element $\omega^{-n}L_q(p + \omega k)$. From (1.2) and (1.3) it follows that the elements along the principal diagonal are given by

$$c_{rr}^{0k}(p) = 1 + \frac{C_n{}^1 p + a_{rr}{}^{0,\,n-1}}{k\omega} + \ldots + \frac{p^n + a_{rr}{}^{0,\,n-1} p^{n-1} + \ldots + a_{rr}{}^{00}}{(k\omega)^n} \tag{2.1}$$

$$(r = 1, \ldots, m, \quad k = 0, \pm 1, \pm 2, \ldots)$$

$$c_{sr}^{qk}(p - q\omega) = \frac{a_{sr}^{q,\,n-1}}{(k-q)\omega} + \frac{C_{n-1}^{1} a_{sr}^{q,\,n-1}\, p + a_{sr}^{q,\,n-2}}{((k-q)\omega)^2} + \ldots + \frac{a_{sr}^{q,\,n-1} p^{n-1} + \ldots + a_{sr}^{q0}}{((k-q)\omega)^n} \tag{2.2}$$

$$(s, r = 1, 2, \ldots, m, \; q = 0, \pm 1, \ldots, \pm l, \; k = 0, \pm 1, \pm 2, \ldots; \; s \neq r \text{ for } q = 0)$$

a) If $a_{sr}^{qk} = 0$ ($s \neq r$ when $q = 0$), then the determinant $\Delta(p)$ may be reduced to the class of normal determinants [3]. One only needs [2] to multiply each column consisting of the elements (2.1) by

$$\exp\{-(C_n^1 p + a_{rr}^{0,\,n-1})/k\omega\} \qquad (k = \pm 1, \pm 2, \ldots, \; r = 1, 2, \ldots, m)$$

This transformation does not alter the value of the determinants $\Delta_\gamma(p)$, and the sum of all the elements of the infinite determinant (not counting the ones along the main diagonal) converges absolutely and uniformly for $p \in \Sigma$. Consequently, $\Delta_\gamma(p)$, and the algebraic complements to the elements on the columns which pass through $L_0(p)$, converge uniformly for $p \in \Sigma$ to functions p which are holomorphic in Σ; and hence all of them are bounded in absolute value by a certain constant.

b) If $a_{sr}^{qk} \neq 0$ ($r \neq s$ when $q = 0$), then one no longer has the absolute convergence of the elements off the main diagonal. Let us order these elements $c_{rr}^{0k}(p)$, $k \neq 0$, which lie along the diagonal as follows: the element $c_{\beta\beta}^{0\alpha}(p)$ precedes $c_{\delta\delta}^{0\gamma}(p)$ provided that $|\alpha| < |\gamma|$; if $|\alpha| = |\gamma|$, then $\alpha > \gamma$; if $\alpha = \gamma$, then $\beta > \delta$.

Consider the diagonal element $c_{rr}^{0k}(p)$ and the elements $c_{sr}^{qk}(p)$ of the column passing through the element $c_{sr}^{0k}(p)$, lying below it for $k > 0$ (and above for $k < 0$) and belonging to the quasi-element

$$((k-q)\,\omega)^{-n} L_q(p+\omega k), \quad k \geqslant 1, \quad k-q \geqslant 1 \quad (k \leqslant 1,\ k-q \leqslant -1)$$

Let us multiply the row containing the element $c_{rr}^{0k}(p)$ by $a_{sr}^{q,n-1}$, and subtract it from the corresponding row containing the element $c_{sr}^{qk}(p)$, and so forth. Let us proceed analogously with the elements in the rows lying to the right for $k > 0$ (to the left for $k < 0$), but only for $k \geqslant 1$, $k - q \geqslant 1$ ($k \leqslant -1$, $k - q \leqslant -1$).

The determinant $\Delta^*(p)$ so obtained may be reduced to the type of determinant considered in (a). In order to do this it is necessary to carry out a countable number of operations, and it may be verified that the determinant $\Delta_\gamma(p)$ may be expressed as a linear combination of a finite number of the minors of the determinant $\Delta_\gamma^*(p)$, and that $\Delta_\gamma(p) \rightrightarrows \Delta^*(p)$ as $\gamma \to \infty$, $p \in \Sigma$. The algebraic complements to the elements lying in the columns passing through $L_0(p)$ in the determinant $\Delta(p)$ may be expressed as linear combinations, with coefficients which are bounded in absolute value, of the corresponding columns in the determinant $\Delta^*(p)$.

Thus, in view of the arbitrariness of the domain Σ, the following theorem has been proved:

Theorem 2.1. The determinant $\Delta(p)$ in (1.8), and the algebraic complements of the elements lying in the columns which pass through $L_0(p)$, converge to entire functions of p which are uniformly bounded in absolute value on each finite dòmain Σ of the complex variable p. For $p \in \Sigma$ the convergence is both absolute and uniform.

Note 2.1. The determinant $\Delta(p)$ and the algebraic complements of the elements in the columns passing through $L_0(p)$ are entire functions of the coefficients a_{sr}^{qj} of (1.3).

Note 2.2. If $\Delta(p_0) = 0$, then $\Delta_\gamma(p)$ will have a zero which is arbitrarily near to p, whenever γ is sufficiently large.

It will be supposed, further, that the domain Σ is so chosen that all the vectors

$$F(p+k\omega), \qquad (k\omega)^{n-1} R(p+k\omega) \qquad (k=0, \pm 1, \pm 2, \ldots)$$

are bounded whenever $p \in \Sigma$. Such a finite domain lies, for example, in $\operatorname{Re} p \geqslant b_1$. Let us suppose that $\Delta(p) > \epsilon > 0$ for $p \in \Sigma$. Let us solve, in succession, the systems of equations arising from (1.5) and (1.7), with determinants $\Delta_\gamma(p)$ as unknowns, by Cramer's rule, it being understood that all other terms in these equations are first transposed to the right-hand sides.

In view of Theorem 2.1, we obtain, as $\gamma \to \infty$, in the domain $\operatorname{Re} p \geqslant b_1$, a solution for $F(p)$ in the form of a certain infinite determinant divided by $\Delta(p)$. Expanding formally this numerator determinant along the column with components $R(p + k\omega)$, we obtain the solution of the difference equations (1.5) in the form

$$F(p) = \sum_{k=-\infty}^{\infty} \Delta^{-1}(p) B_k(p) R(p+\omega k) \tag{2.3}$$

where the elements of the matrices $B_k(p)$ are entire functions of p. From the properties of the solutions of system (1.1) it may be asserted that the series (2.3) converges for $\operatorname{Re} p \geqslant b_1$, and that it indeed represents a solution of (1.5).

An approximate solution of the system (1.1) may be obtained by finding the inverse Laplace transform of an approximate solution for $F(p)$ whose components are the ratios of finite determinants. In this way one may seek a solution of the nonhomogeneous system (1.1) subject to initial conditions.

3. Let us employ the methods of [4] to obtain another representation for the determinant $\Delta(p)$. Let us factor out the terms along the main diagonal of the determinant $\Delta(p)$. Let $\rho_{sh}(s = 1, \ldots, m,\ h = 1, \ldots, n)$ be the roots of the equations $a_{ss}^{00}(p) = 0$. Then

$$\prod_{k=-\infty}^{\infty} \prod_{s=1}^{m} (k\omega)^{-n} a_{ss}^{00}(p+\omega k) = \left(\frac{\omega}{\pi}\right)^{mn} \prod_{s=1}^{m} \prod_{h=1}^{n} \sin\frac{\pi}{\omega}(p-\rho_{sh}) \tag{3.1}$$

Let us call the remaining determinant $D(p)$. Since the product (3.1) of the diagonal elements is convergent, it follows that the determinant $D(p)$ converges at all numbers which differ from the numbers ρ_{shk}:

$$\rho_{shk} = \rho_{sh} - k\omega \qquad (s = 1, \ldots, m,\ h = 1, \ldots, n,\ k = 0, \pm 1, \pm 2, \ldots) \tag{3.2}$$

The determinant $D(p)$ is a periodic meromorphic function of p with period ω. Let us reduce it to normal form by the method of Section 2. Then

$$D(p) \rightrightarrows 1 \qquad \text{for} \quad |\operatorname{Re} p| \to \infty \tag{3.3}$$

Suppose, for simplicity, that the numbers ρ_{shk} are all distinct (the case where some of them coincide is considered in [2]) and we obtain, from [4]:

$$D(p) = 1 + \sum_{s=1}^{m} \sum_{h=1}^{n} \delta_{sh} \cot\left(\frac{\pi}{\omega}(p - \rho_{sh})\right), \qquad \sum_{s=1}^{m} \sum_{h=1}^{n} \delta_{sh} = 0 \tag{3.4}$$

where the δ_{sh} are certain constants. Let us express $\Delta(p)$ in terms of $D(p)$ and the product (3.1), setting, in (3.1) and (3.4)

$$z = \exp\left(\frac{2\pi i}{\omega} p\right), \quad z_{sh} = \exp\left(\frac{2\pi i}{\omega} p_{sh}\right) \qquad (s = 1, \ldots, m,\ h = 1, \ldots, p) \tag{3.5}$$

Then

$$\Delta\left(\frac{\omega}{2\pi i} \ln z\right) = \left(\frac{\omega}{2\pi i}\right)^{mn} \left(\prod_{s=1}^{m} \prod_{h=1}^{n} (z z_{sh})^{-\frac{1}{2}}\right) \{(z - z_{11})(z - z_{12}) \ldots (z - z_{mn}) + \\ + i\delta_{11}(z + z_{11})(z - z_{12}) \ldots (z - z_{mn}) + \ldots + i\delta_{mn}(z - z_{11})(z - z_{12}) \ldots (z + z_{mn})\} \tag{3.6}$$

From (3.4) it follows that the braces contain a polynomial with the leading term z^{mn} and the constant term $z_{11}z_{12}\ldots z_{mn}$. Equating this determinant to zero we obtain the characteristic equation for the multipliers of the solutions of system (1.1). The factor in front of the braces in (3.6) does not vanish for any finite value of p. From this it follows that the Hill determinant $\Delta(p)$ of (1.8), times a certain known multiple, may be expressed directly in terms of the coefficients of the characteristic polynomial of the solutions of the system (1.1), due account being taken of (3.5). The theorem is proved.

Theorem 3.1. The solution of the system of linear difference equations (1.5), which is bounded for Re $p \geqslant b_1$, (b_1 = const), is uniquely representable in the form (2.3), where the elements of the matrices $B_k(p)$ are entire functions of p, and $\Delta(p)$ is an entire function of p with period ω, whose zeros are the characteristic exponents of the solutions of the system of equations (1.1).

The last assertion of Theorem 3.1 has been known for a long time [1, 2]. In these papers, and also in [5, 7], representations of the form (3.6), or their modifications, are used to determine the constants δ_{sh}.

In [6] the determinant $\Delta(p)$ is used for the solution of the equation $\Delta(p) = 0$. In Section 5 of the present paper a method related to that of [6] is used.

4. Consider the linear differential equation with sinusoidal coefficients [10]

$$\sum_{k=0}^{n} (a_k^{(0)} + a_k^{(1)} e^{-\omega t} + a_k^{(-1)} e^{\omega t}) \frac{d^k y}{dt^k} = \varphi(t) \tag{4.1}$$

where $a_k^{(j)}$ are complex constants; $\phi(t) \risingdotseq g(p)$, where $g(p)$ is a regular and bounded function of p for Re $p \geqslant b$, Re $\omega = 0$. Let us seek the transform $f(p)$ of the solution $y(t)$ $(t \geqslant 0)$ with initial conditions

$$y(0) = y_0^{(0)}, \ldots, y^{(n-1)}(0) = y_0^{(n-1)} \tag{4.2}$$

Let us assume that

$$|a_n^{(0)}|^2 > 4 |a_n^{(1)} a_n^{(-1)}| \tag{4.3}$$

that is, the coefficient of the highest-order derivative does not vanish for any t. Let us introduce the notations

$$f_q(p) = \sum_{k=0}^{n} a_k^{(q)} p^k, \qquad \psi_q(p) = \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} a_k^{(q)} y_0^{(j)} p^{k-j-1} \qquad (q = 0, \pm 1) \tag{4.4}$$

The difference equation for $f(p)$ takes the form

$$\sum_{q=-1}^{1} f_q(p + \omega q) f(p + \omega q) = g(p) + \sum_{q=-1}^{1} \psi_q(p + q\omega) \equiv r(p) \tag{4.5}$$

Forming the ratio of infinite determinants, and expanding them (see Section 2), we obtain the solution of (4.1) as a series

$$\begin{aligned} f(p) = \frac{1}{\nabla(p)} \Big\{ r(p) - \frac{f_1(p+\omega)}{f_0(p+\omega) - s(p+\omega)} \Big(r(p+\omega) - \\ - \frac{f_1(p+2\omega)}{f_0(p+2\omega) - s(p+2\omega)} (r(p+2\omega) - \ldots) \Big) - \\ - \frac{f_{-1}(p-\omega)}{f_0(p-\omega) - h(p-\omega)} \Big(r(p-\omega) - \frac{f_{-1}(p-2\omega)}{f_0(p-2\omega) - h(p-2\omega)} (r(p-2\omega) - \ldots) \Big) \Big\} \end{aligned} \tag{4.6}$$

where $s(p)$ and $h(p)$ are given by the continued fractions [8]:

$$s(p) = \cfrac{f_1(p+\omega) f_{-1}(p)}{f_0(p+\omega) - \cfrac{f_1(p+2\omega) f_{-1}(p+\omega)}{f_0(p+2\omega) - \cfrac{f_1(p-3\omega) f_{-1}(p-2\omega)}{f_0(p-3\omega) - \ldots}}} \tag{4.7}$$

$$h(p)=\cfrac{f_{-1}(p-\omega)f_1(p)}{f_0(p-\omega)-\cfrac{f_{-1}(p-2\omega)f_1(p-\omega)}{f_0(p-2\omega)-\cfrac{f_{-1}(p-3\omega)f_1(p-2\omega)}{f_0(p-3\omega)-\dots}}}$$

In view of (4.3), the continued fractions (4.7) converge for all values of p to meromorphic functions [8,10]. The representation of the transform $f(p)$ of the solution $y(t)$ by means of (4.6) and (4.7) is very convenient for the numerical computation of the solution of Equation (4.1) in the special case when $\phi(t)$ is a sum of terms of the form $c_j t^{\nu_j} e^{\omega_j t}$. It should be noticed that all roots of the equation $\nabla(p) = 0$ are characteristic exponents of solutions of (4.1). The converse is not always true.

The equation for the characteristic exponents of the solutions, in the form

$$\nabla(p)\equiv f_0(p)-s(p)-h(p)=0 \tag{4.8}$$

was obtained by Ince [11] for Mathieu's equation, and for Equation (4.1) by Patry [10], as a result of expanding the solution $y(t)$ of (4.1) in a Fourier series.

Example 1. [12]. Consider Mathieu's equation in the presence of friction

$$\frac{d^2y}{dt^2}+c\,\frac{dy}{dt}+(a+2b\cos 2t)\,y=0 \quad (c\geqslant 0) \tag{4.9}$$

The characteristic exponents of the solutions of Equation (4.9) are obtained from the equation

$$\nabla(p)\equiv f_0(p)-\cfrac{b^2}{f_0(p+2i)-\cfrac{b^2}{f_0(p+4i)-\dots}}-\cfrac{b^2}{f_0(p-2i)-\cfrac{b^2}{f_0(p-4i)-\dots}}=0 \tag{4.10}$$

where $f_0(p) = p^2 + cp + a$. Using (4.10), let us write the equation of the boundary of the γth domain of instability of the solutions of Equation (4.9). Since at the boundary of the domain of instability one of the characteristic exponents equals γi, where γ is an integer, the desired equation is $\nabla(\gamma i) = 0$. In particular, for $\gamma = 1$ we obtain

$$\left|a-1+ci-\cfrac{b^2}{a-9+3ci-\cfrac{b^2}{a-25+5ci-\dots}}\right|^2=b^2 \tag{4.11}$$

Equation (4.10) holds for finite a, b, c; it is also the equation of all odd domains of instability.

5. Consider the system of differential equations of the form

$$\frac{d^2Y}{dt^2} + \mu N(\theta t)\frac{dY}{dt} + (C + \mu P(\theta t))Y = 0 \tag{5.1}$$

where $C = (\omega_1{}^2, \ldots, \omega_m{}^2)$ is a diagonal matrix, $\theta > 0$, $\omega_j{}^2 > 0$ $(j = 1, \ldots, m)$, and μ is a small parameter:

$$N(\tau) = \sum_{k=-l}^{l} N^{(k)} e^{ik\tau}, \qquad N^{(k)} = \| \nu_{js}{}^{(k)} \|_1^m$$
$$P(\tau) = \sum_{k=-l}^{l} P^{(k)} e^{ik\tau}, \qquad P^{(k)} = \| \pi_{js}{}^{(k)} \|_1^m \qquad (k = 0, \pm 1, \ldots, \pm l) \tag{5.2}$$

with complex constants $\nu_{js}{}^{(k)}$, $\pi_{js}{}^{(k)}$, and l is a finite number.

Let us assume that $\theta = \theta_0 + \mu\lambda$, $(\lambda = \text{const})$, $\theta_0 > 0$, and let us apply Theorem 2.1, in order to obtain the characteristic exponents of the solutions of Equation (5.1), as functions of the parameter μ. Consider the general case of resonance and the set of $2m$ numbers

$$\omega_1, \ldots, \omega_m, -\omega_1, \ldots, -\omega_m \tag{5.3}$$

Let us divide them into two groups, putting in one group all the numbers (5.3) which differ among themselves by quantities $k\theta_0$, where k is an integer. These groups are

$$(\rho_{11}, \ldots, \rho_{1\beta_1}), \ldots, (\rho_{\alpha 1}, \ldots, \rho_{\alpha\beta_\alpha}) \qquad (\beta_2 + \ldots + \beta_\alpha = 2m) \tag{5.4}$$

Let ρ_{jh} indicate one of the numbers (5.3); if this is the number ω_s or ω_{-s}, then the symbol $[jh]$ will indicate number s, where j designates the group number and h the number within a group.

Let us suppose that the ρ_{jh} are arranged in nonincreasing order for $h = 1, \ldots, \beta_j$, and let us denote

$$k_{jh} = (\rho_{j1} - \rho_{jh})\theta_0^{-1} \qquad (j = 1, \ldots, \alpha,\ h = 1, \ldots, \beta_j) \tag{5.5}$$

where the numbers k_{jh} are nonnegative integers satisfying $0 = k_{j1} \leqslant \cdots \leqslant k_{j\beta_j}$.

Let us set $p = i\rho_{j1} + i\mu z$, for $j = 1, \ldots, \alpha$, in the determinant $\Delta(p)$ of (1.8). Since for Equation (5.1) we have (see Section 1)

$$L_0(p) = Ep^2 + \mu N^{(0)}p + C + \mu P^{(0)}, \quad L_q(p) = \mu N^{(-q)}p + \mu P^{(-q)} \ (q \neq 0) \tag{5.6}$$

then along the main diagonal there are terms of order μ, and off the main diagonal all terms are of order μ.

Let us make use of Note 2.2 of Theorem 2.1. In order to do this, consider a sufficiently large determinant $\Delta_\gamma(p)$ (see Section 2). It may be regarded as an approximation to $\Delta(p)$. Regarding the equation $\Delta_\gamma(i\rho_{j1} + i\mu z) = 0$ as an equation defining the algebraic function $z(\mu)$, we obtain from Newton's diagram [9] that the characteristic exponents of the solutions of system (5.1), $p_{j\xi}(\mu)$, $(j = 1, \ldots, \alpha;\ \xi = 1, \ldots, \beta_j)$ may be expanded in series of the form

$$p_{j\xi}(\mu) = i\rho_{j1} + i\mu\left(z_{j\xi} + \mu^{\sigma_{j\xi}}(\ldots)\ldots\right) \tag{5.7}$$

where $\sigma_{j\xi} > 0$ in general fractional numbers, and the numbers $z_{j\xi}$, for $j = 1, \ldots, \alpha;\ \xi = 1, \ldots, \beta_j$, are roots of the equations

$$\operatorname{Det} \| b_{sr}^{(j)} + \delta_{sr} k_{js}\lambda - \delta_{sr} z \|_1^{\beta j} = 0 \quad (j = 1, \ldots, \alpha) \tag{5.8}$$

with

$$b_{sr}{}^{(j)} = \frac{i\sqrt{\rho_{jr}}}{2\sqrt{\rho_{js}}}\,\nu_{[js][jr]}^{(j,r,s)} + \frac{1}{2\sqrt{\rho_{js}}\sqrt{\rho_{jr}}}\,\pi_{[js][jr]}^{(j,r,s)} \tag{5.9}$$

$$((j, r, s) \equiv k_{jr} - k_{js},\ \delta_{sr} - \text{Kronecker's symbol}$$

For the proof, let us mark the columns and rows which contain elements along the main diagonal of the determinant $\Delta_\gamma(i\rho_{j1} + i\mu z)$ which vanish for $\mu = 0$. Consider the determinant formed by the intersection of these columns and rows. This determinant differs only by an unessential factor from the determinant (5.8). The remaining elements need not be taken into account in a first approximation.

Knowing the numbers $z_{j\xi}$, we may seek to determine the stability of system (5.1) in first approximation. Upon varying λ we may determine in the plane $\theta\mu$ wide domains of parametric resonance which are adjacent to the frequency θ_0. It should be noticed that the finiteness of l, γ does not influence the final formulas (5.8), (5.9). Consequently, the functions $N(\tau)$, $P(\tau)$ may be regarded as of integrable square on the interval $[0, 2\pi]$.

Formulas analogous to (5.8), (5.9), for the canonical case of system (5.1), have already been obtained by Iakubovich [14].

Example 2. Consider the system of equations

$$\frac{d^2y_1}{dt^2} + \mu\left(\varepsilon_{11}\frac{dy_1}{dt} + \varepsilon_{12}\frac{dy_2}{dt}\right) + \omega_1^2 y_1 + 2\mu\,(a_{11}y_1 + a_{12}y_2)\cos\theta t = 0 \tag{5.10}$$

$$\frac{d^2y_2}{dt^2} + \mu\left(\varepsilon_{21}\frac{dy_1}{dt} + \varepsilon_{22}\frac{dy_2}{dt}\right) + \omega_2^2 y_2 + 2\mu\,(a_{21}y_1 + a_{22}y_2)\cos\theta t = 0$$

where $\epsilon_{11}, \ldots, a_{22}$ are real numbers, and $\theta_0 = \omega_1 + \omega_2$, $\omega_1 > \omega_2 > 0$.

In the notation of Section 5 we have

$$\rho_{11} = \omega_1, \quad \rho_{12} = -\omega_2, \quad \rho_{21} = \omega_2, \quad \rho_{22} = -\omega_1, \quad [11] = [22] = 1, \quad [12] = [21] = 2$$

$$k_{11} = 0, \ k_{12} = 1, \ k_{21} = 0, \ k_{22} = 1$$

Equation (5.8) for z_{11}, z_{12} takes the form

$$\begin{vmatrix} \frac{i}{2}\varepsilon_{11} - z & \frac{a_{12}}{2\sqrt{-\omega_1\omega_2}} \\ \frac{a_{21}}{2\sqrt{-\omega_1\omega_2}} & \frac{i}{2}\varepsilon_{22} + \lambda - z \end{vmatrix} = 0 \tag{5.11}$$

If we require that the inequalities $\operatorname{Im} z_{11} > 0$, $\operatorname{Im} z_{12} > 0$ hold for every $\lambda \in (-\infty, \infty)$, we obtain the conditions

$$\varepsilon_{11} > 0, \qquad \varepsilon_{22} > 0, \qquad \varepsilon_{11}\varepsilon_{22}\omega_1\omega_2 > a_{12}a_{21} \tag{5.12}$$

In view of the fact that the coefficients of system (5.10) are real, $\operatorname{Im} z_{21} > 0$, $\operatorname{Im} z_{22} > 0$; therefore conditions (5.12) are the conditions for the asymptotic stability of the solutions of (5.10) for small values of μ.

In the equation $\Delta(p) = 0$ let us make the following simplifying transformation: let us divide the rows of the determinant by the elements of the rows which lie on the principal diagonal of the determinant $\Delta(p)$, with the exception of the rows in which these elements vanish for $\mu = 0$, $p = \rho_{j1} i$. If we expand the determinant so obtained in terms of the first order in $p = \rho_{ji} + \mu z i$, we must first seek the elements which lie on the intersection of columns and rows passing through elements which lie on the principal diagonal of $\Delta(\rho_{j1} i + \mu z i)$ and vanish for $\mu = 0$. In the following higher-order terms there appear only elements lying in the already-mentioned rows and columns. In particular, if in (5.1) the matrix $P(\theta t)$ is self adjoint, then

$$N(\theta t) \equiv 0, \ \mu = 1, \ l = \infty, \ \beta_1 = 2, \ \rho_{11} = \omega_g, \ \rho_{12} = -\omega_h, \ \theta_0 = \frac{\omega_g + \omega_h}{k}$$

(i.e. the relation $\omega_g + \omega_h = k\theta_0$ holds only for the given g, h, k), we obtain an equation for p, written below, including third-order terms in $\pi_{js}^{(k)}$:

$$(p^2 + \omega_g^2 + \pi_{gg}^{(0)})((p - k\theta i)^2 + \omega_h^2 + \pi_{hh}^{(0)}) - \pi_{gh}^{(k)}\pi_{hg}^{(-k)} - \tag{5.13}$$
$$- (p^2 + \omega_g^2 + \pi_{gg}^{(0)}) \sum_{r=1}^{m} \sum_{j=-\infty}^{\infty}{}' \frac{\pi_{rh}^{(-j)}\pi_{hr}^{(j)}}{\omega_r^2 - (\omega_h + j\theta_0)^2} + \pi_{gh}^{(k)} \sum_{r=1}^{m} \sum_{j=-\infty}^{\infty}{}'$$

$$\frac{\pi_{rg}^{(-k-j)}\pi_{hr}^{(j)}}{\omega_r^2-(\omega_h+j\theta_0)^2} - ((p-k\theta i)^2+\omega_h^2+\pi_{hh}^{(0)}\sum_{r=1}^{m}\sum_{j=-\infty}^{\infty}{}' \frac{\pi_{rg}^{(j)}\pi_{hr}^{(-j)}}{\omega_r^2-(\omega_g+j\theta_0)^2} +$$
$$+\pi_{hg}^{(-k)}\sum_{r=1}^{m}\sum_{j=-\infty}^{\infty}{}' \frac{\pi_{rh}^{(k+j)}\pi_{gr}^{(-j)}}{\omega_r^2-(\omega_g+j\theta_0)^2} = 0$$

where the prime indicates that the terms in the summations with zero denominators are omitted.

If $\pi_{gh}^{(k)}\pi_{hg}^{(-k)} \neq 0$, then Equation (5.13) enables one to determine two characteristic exponents which are near $i\omega_g$ up to second-order terms.

By requiring that Equation (5.13) have a multiple root near $p = i\omega_g$, we obtain the equation of the boundary of the domain of stability, up to terms of second order with respect to $\pi_{js}^{(k)}$.

Example 3. [13]. Consider the equation

$$\frac{d^2Y}{dt^2} + [P_0^2 - \varphi N]\,Y = 0 \tag{5.14}$$

where

$$P_0 = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \varphi = \alpha + \beta\cos\theta t, \qquad \omega_1 > \omega_2 > 0$$
$$\theta_0 = \omega_1 + \omega_2, \qquad k = 1 \qquad \pi_{12}^{(0)} = \pi_{22}^{(0)} = -\alpha, \qquad \pi_{12}^{(1)} = \pi_{21}^{(1)} = \pi_{12}^{(-1)} = \pi_{21}^{(-1)} = -\frac{\beta}{2}$$

we obtain that Equation (5.13), constructed for Equation (5.14), has a multiple root provided that

$$\theta_\pm - \theta_0 = \pm\frac{\beta}{2\sqrt{\omega_1\omega_2}} - \frac{1}{16\theta_0\omega_1\omega_2}(8\alpha^2+\beta^2) + \ldots \tag{5.15}$$

For $\theta_- < \theta < \theta_+$, the characteristic exponents have a nonzero real part, and the solutions of Equation (5.14) are unstable [13].

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